TM5 PR 9.58

CONSIDER A SINGLE STAGE ROCKET TAKING OFF FROM THE EARTH
SHOW THAT THE HEIGHT AT BURNOUT IS

$$y_b = ut_b - \frac{1}{2}gt_b^2 - \frac{mu}{\alpha}\ln\left(\frac{m_0}{m}\right)$$

AND FIND HOW MUCH HIGHER IT WILL GO.

FOR ASCENT UNDER GRAVITY,

$$v = \frac{dy}{dt} = -gt + u\ln\left(\frac{m_0}{m}\right)$$

THUS

$$\int_0^{y_b} dy = \int_0^{t_b}\left[-gt + u\ln\left(\frac{m_0}{m}\right)\right]dt$$

ASSUMING A CONSTANT BURN RATE

$$\frac{dm}{dt} = -\alpha \quad \Rightarrow \quad dt = -\frac{1}{\alpha}dm$$

CHANGE VARIABLE IN THE SECOND INTEGRAL

$$y_b = -\frac{1}{2}gt_b^2 - \frac{u}{a}\int_{m_0}^{m}\ln\left(\frac{m_0}{m}\right)dm$$

OR

$$y_b = -\frac{1}{2}gt_b^2 + \frac{u}{\alpha}\int_{m_0}^{m}\ln\left(\frac{m}{m_0}\right)dm$$

INTEGRAL # 299 GIVES $\int \ln(x) = x\ln(x) - x$, THUS

$$y_b = -\frac{1}{2}gt_b^2 + \frac{u}{\alpha}\left[m\ln\left(\frac{m}{m_0}\right) - m\right]_{m_0}^{m}$$

$$= -\frac{1}{2}gt_b^2 + \frac{u}{\alpha}\left[m\ln\left(\frac{m}{m_0}\right) - m - m_0\cancel{\ln\left(\frac{m_0}{m_0}\right)}^{0} + m_0\right]$$

$$= -\frac{1}{2}gt_b^2 + \frac{u}{\alpha}\left[m\ln\left(\frac{m}{m_0}\right) - (m - m_0)\right]$$

FROM THE CONSTANT BURN RATE

$$\int_0^{t_b} dt = -\frac{1}{\alpha}\int_{m_0}^{m} dm = -\frac{m - m_0}{\alpha} \Rightarrow m_0 - m = \alpha t_b$$

NOTE BOOK

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THUS, FROM

$$y_b = -\frac{1}{2}g t_b^2 + \frac{um}{\alpha}\ln\left(\frac{m}{m_0}\right) + \frac{u(m_0 - m)}{\alpha}$$

WE GET

$$y_b = -\frac{1}{2}g t_b^2 + \frac{um}{\alpha}\ln\left(\frac{m}{m_0}\right) + \frac{u}{\cancel{\alpha}}(\cancel{\alpha} t_b)$$

$$\Rightarrow \boxed{y_b = ut_b - \frac{1}{2}g t_b^2 + \frac{um}{\alpha}\ln\left(\frac{m}{m_0}\right)} \quad \text{QED!}$$

HOW MUCH HIGHER WILL IT GO?

- NOW ITS A PROJECTILE WITH

$$v_{0,p} = v_{b,\text{ROCKET}} = -g t_b + u\ln\left(\frac{m_0}{m}\right)$$

AT THE TOP, 0 ... 0 TO FIND EXTRA HEIGHT

$$\cancel{v^2} = v_{0,p}^2 - 2g(y_{TOP} - \cancel{y_0})$$

$$\Rightarrow \quad y_{TOP} = \frac{v_{0p}^2}{2g}$$

OR

$$\boxed{y_{TOP} = \frac{1}{2g}\left[-g t_b + u\ln\left(\frac{m_0}{m}\right)\right]^2}$$